Hölder properties of perturbed skew products and Fubini regained*

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Abstract

In 2006 A. Gorodetski proved that central fibers of perturbed skew products are Hölder continuous with respect to the base point. In the present paper we give an explicit estimate of the Hölder exponent mentioned above. Moreover, we extend the Gorodetski theorem from the case when the fiber maps are close to the identity to a much wider class that satisfy the so-called modified dominated splitting condition. In many cases (for example, in the case of skew products over the solenoid or over linear Anosov diffeomorphisms of a torus), the Hölder exponent is close to 1. This allows us in a sense to overcome the so called Fubini nightmare. Namely, we prove that the union of central fibers that are strongly atypical from the point of view of the ergodic theory, has Lebesgue measure zero, despite the lack of absolute continuity of the holonomy map for the central foliation. For that we revisit the Hirsch-Pugh-Shub theory, and estimate the contraction constant of the graph transform map.

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1 Introduction

1.1 Skew products and Hölder continuity

In this paper we study perturbations of skew products over hyperbolic maps in the base. Under the so-called dominated splitting condition, these perturbations have an invariant center lamination (see [9]). It was proven by Gorodetski that this lamination will be Hölder continuous, [8] (see also [18] for an earlier particular result). In this paper we estimate the Hölder exponent, and prove that in some cases it can be made arbitrarily close to 1 by making the perturbation small enough.

Such a center lamination allows us to conjugate any perturbation of a skew product with another skew product, which can be very useful in applications. A priori, the conjugation is only a homeomorphism, and as such it might not behave nicely with the measure. However, the Hölder property which we prove gives us some control over this conjugation, and allows to resolve a number of measure-related issues.

For example, the Hölder property allows us to overcome in some sense the Fubini nightmare. In more detail, we prove that perturbations of a skew product over the Smale-Williams solenoid are semiconjugated with the duplication of the circle. We prove that the fibers of this semiconjugacy q (i.e. the manifolds $q^{-1}(y),\ y \in S^1$) are Hölder in y with exponent close to 1. As a corollary, we prove that for a set $A \subset S^1$ of points with "strongly nonergodic orbits" under the duplication of the circle, the inverse image $q^{-1}(A)$ has Lebesgue measure 0 (Theorem 3 below). This is proved despite the fact that the foliation by the fibers $q^{-1}(y), y \in S^1$ may not have an absolutely continuous holonomy.

The entire Section 1 is concerned with statements of results. In Subsection 1.2 below, we introduce our Main Theorem 1 for perturbations of skew products over arbitrary hyperbolic maps. In the next Subsection 1.3 we introduce Theorem 2, which improves the Main Theorem in some cases (an important example of which being the solenoid). Two applications of Theorem 2 are presented in Subsections 1.4 and 1.5.

The sections after that will be concerned with proofs. In Sections 2 and 3

we work with laminations and graph transform operators, culminating with the proof of the Main Theorem 1. Several of the results we obtain help us in Section 4, where we prove Theorem 2. In Section 5, we *regain the Fubini property of our central leaves*, thus proving Theorem 3. Finally, Section 6 consists of an appendix where several technical results are proved.

1.2 Persistence and Hölder property for skew products

Throughout this paper a C^r-*morphism* will refer to a C^r map with a C^r inverse. We will use this notion both for maps of a manifold (with or without boundary) onto itself, and for maps of a manifold with boundary strictly into itself.

Given a linear operator A on a normed linear space V, when we write $a \leq |A| \leq b$ we mean that

$$a|v| \leq |A(v)| \leq b|v|, \quad \forall v \in V.$$

We will use this convention repeatedly throughout the paper.

Let B be a compact Riemannian manifold, henceforth called the *base*. Suppose $h : B \to B$ is a C^2-morphism with a hyperbolic invariant subset $\Lambda \subset B$. When h is onto, we can take $B = \Lambda$. When h is into, we can take Λ to be the maximal attractor of h:

$$\Lambda = \bigcap_{n \geq 0} h^n(B).$$

Being hyperbolic, the map h will have contracting and expanding directions in the vicinity of Λ. Thus there exist a Riemannian metric d on B and real numbers $0 \leq \lambda_- \leq \lambda < 1$ and $0 \leq \mu_- \leq \mu < 1$, as well as a decomposition of the tangent bundle:

$$TB|_\Lambda = E^s \oplus E^u, \tag{1}$$

such that

$$dh : E^s \to E^s \quad \text{and} \quad \lambda_- \leq |dh| \leq \lambda,$$
$$dh : E^u \to E^u \quad \text{and} \quad \mu_- \leq |dh^{-1}| \leq \mu. \tag{2}$$

Note that if $\lambda_- = \mu_- = 0$, then we get the standard notion of hyperbolicity.

We assume that the bundles E^s and E^u are trivialized, i.e. that there exist isomorphisms over B:

$$\varphi^s : B \times \mathbb{R}^k \to E^s, \quad \varphi^u : B \times \mathbb{R}^l \to E^u \tag{3}$$

for some positive integers $k, l \geq 0$. The above is a technical condition necessary for our proof, but we conjecture that all our results hold without it. Let us note that it holds when h is the Smale-Williams solenoid map or any linear Anosov diffeomorphism of a torus.

Definition 1 *An invariant set Λ of a map h with the above properties will be called $(\lambda_-, \lambda, \mu_-, \mu)$- hyperbolic.*

Take another compact manifold M, called the *fiber*, and form the Cartesian product $X = B \times M$. A *skew product* over h is defined as any C^1-map of the form

$$\mathcal{F} : X \to X, \quad \mathcal{F}(b, m) = (h(b), f_b(m)), \tag{4}$$

where $f_b(m) : M \to M$ is a C^1 family of C^1-morphisms.

Definition 2 *We say that the skew product* (4) *satisfies the* modified dominated splitting condition *if*

$$\max\left(\max(\lambda, \mu) + \left|\left|\frac{\partial f_b^{\pm 1}}{\partial b}\right|\right|_{C^0(X)}, \left|\left|\frac{\partial f_b^{\pm 1}}{\partial m}\right|\right|_{C^0(X)}\right) =: L < \min(\lambda^{-1}, \mu^{-1}). \tag{5}$$

Skew products are very useful in constructing dynamical systems with various properties. However, one often wants to study generic phenomena of dynamical systems, and therefore one also has to study small perturbations of skew products.

Definition 3 *Given $\rho > 0$, a $\rho-$*perturbation *of the skew product* (4) *is a C^1-morphism $\mathcal{G} : X \to X$ such that*

$$d(\mathcal{G}^{\pm 1}, \mathcal{F}^{\pm 1})_{C^1(X)} < \rho. \tag{6}$$

Let us make a notational convention. In this paper, we will consider a fixed skew product $\mathcal{F}$ and a neighborhood $\Omega \ni \mathcal{F}$ in the C^1-norm. We will often be concerned with small perturbations $\mathcal{G} \in \Omega$ of $\mathcal{F}$, and with various geometric objects related to these perturbations (such as central foliations, Hölder exponents etc). The leaves of central foliations of the perturbed maps $\mathcal{G}$ are graphs of parameter dependent maps β_b, or in other words, parameter dependent perturbations of the central fibers of $\mathcal{F}$. Whenever we write $||\beta_b|| = O(\rho)$, we mean that there exists a constant C depending only on Ω, such that for any $\rho-$perturbation $\mathcal{G} \in \Omega$ the maps corresponding to all central leaves satisfy the inequality $||\beta_b|| \leq C\rho$. Thus the operator that maps $\mathcal{G}$ to β_b is Lipshitz at $\mathcal{F}$ with constant C (uniformly in b). We will consider other (parameter dependent) operators and functionals defined on Ω; the expression $O(\rho)$ has the same meaning for them.

Small perturbations of skew products are not necessarily skew products anymore. However, in this paper we will show that they are conjugated to skew products, and moreover the conjugation map satisfies a Hölder continuity property.

We will now state our main result.

Theorem 1 (The Main Theorem) *Consider a skew product $\mathcal{F}$ as in* (4) *over a $(\lambda_-, \lambda, \mu_-, \mu)-$ hyperbolic map in the base, satisfying the modified dominated splitting condition. Then for small enough $\rho > 0$, any $\rho-$ perturbation $\mathcal{G}$ of $\mathcal{F}$ has the following properties:*

a) There exists a $\mathcal{G}-$invariant set $Y \subset X$ and a continuous map $p : Y \to B$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\mathcal{G}} & Y \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \Lambda & \xrightarrow{h} & \Lambda \end{array} \tag{7}$$

commutes. Moreover, the map

$$H : Y \to \Lambda \times M, \quad H(b, m) = (p(b, m), m) \tag{8}$$

is a homeomorphism.

b) The fibers $W_b = p^{-1}(b)$ are Lipschitz close to vertical (constant) fibers, and Hölder continuous in b. This means that W_b is the graph of a Lipschitz map $\widetilde{\beta}_b : M \to B$ such that

$$d(\widetilde{\beta}_b, b)_{C^0} \leq O(\rho), \quad Lip\ \widetilde{\beta}_b \leq O(\rho) \tag{9}$$

$$d(\widetilde{\beta}_b, \widetilde{\beta}_{b'})_{C^0} \leq \frac{d(b, b')^{\alpha - O(\rho)}}{O(\rho)^\alpha}, \tag{10}$$

where

$$\alpha = \min\left(\frac{\ln \lambda}{\ln \lambda_-}, \frac{\ln \mu}{\ln \mu_-}\right). \tag{11}$$

Moreover, the map H^{-1} is also Hölder continuous, with the same α.

Remark 1 *Let us first make a remark about the exponent α. In many cases (e.g. when h is the solenoid map or a linear Anosov diffeomorphism of a torus), it may happen that $\lambda_- = \lambda$ and $\mu_- = \mu$. In that case, in the above theorem we have $\alpha = 1$, and thus the Hölder exponent can be made arbitrarily close to 1 by making ρ small enough.*

Remark 2 *If $\Lambda = B$ (which would require h to be surjective) the invariant set Y equals the entire phase space X. This may be proven in similar fashion to Proposition 3 below.*

Let us explain the usefulness of this Theorem. Quite often, one may use skew products $\mathcal{F}$ to exhibit various dynamical or ergodic phenomena (see [5], [6], [7], [11], [3]). One would like to prove the same properties for small perturbations $\mathcal{G}$ of $\mathcal{F}$, but $\mathcal{G}$ is a priori not a skew product anymore. However, letting $G = H \circ \mathcal{G}|_Y \circ H^{-1}$, statement a) of the above theorem implies that $G : \Lambda \times M \to \Lambda \times M$ is indeed a skew product:

$$G(b, m) = (h(b), g_b(m)).$$

One can then study the dynamical properties of the more mysterious map $\mathcal{G}|_Y$ by studying the dynamical properties of its conjugate skew product G.

The fiber maps g_b of the skew product G are C^1-close to those of the skew product $\mathcal{F}|_Y$, in the following sense:

$$d(g_b^{\pm 1}, f_b^{\pm 1})_{C^1} \leq O(\rho). \tag{12}$$

But what can be said about the fiber maps g_b for different b's? Since $\mathcal{F}$ is a C^1-morphism, the fiber maps f_b depend in a C^1 fashion on the point $b \in \Lambda$. Such a result fails for the fiber maps g_b, but statement b) of Theorem 1 implies that the fiber maps g_b depend Hölder continuously on the point $b \in \Lambda$:

$$d(g_b^{\pm 1}, g_{b'}^{\pm 1})_{C^0} \leq O(d(b, b')^\alpha), \tag{13}$$

where α is given by (11). A skew product G whose fiber maps satisfy (13) will be called a *Hölder skew product*. Thus Theorem 1 can be summarized as follows:

Let $\mathcal{G}$ be any small perturbation of a C^2 skew product $\mathcal{F}$ over a $(\lambda_-, \lambda, \mu_-, \mu)-$ hyperbolic map h, satisfying the modified dominated splitting condition (5). Then $\mathcal{G}$ has an invariant set Y such that the restriction of $\mathcal{G}|Y$ is conjugated to a Hölder skew product close to $\mathcal{F}|\Lambda \times M$, in the sense of (11), (12) and (13).

1.3 The solenoid case

In this section we will present a partial improvement of Theorem 1 that is inspired by the example of the Smale-Williams solenoid. Let us begin by introducing and describing the solenoid map. Fix constants $R \geq 2$ and $\lambda < 0.1$, whose particular values will not be important. Let B denote the solid torus

$$B = S^1 \times D, \text{ where } S^1 = \{y \in \mathbb{R}/\mathbb{Z}\}, \ D = \{z \in \mathbb{C} | |z| \leq R\}.$$

The *solenoid map* is defined as

$$h : B \to B, \quad h(y, z) = (2y, e^{2\pi i y} + \lambda z). \tag{14}$$

The maximal attractor of the solenoid map:

$$\Lambda = \bigcap_{k=0}^{\infty} h^k(B)$$

is called the *Smale-Williams solenoid*. It is a hyperbolic invariant set with contraction coefficient λ and expansion coefficient $\mu^{-1} = 2$ (we take the sup norm in T_bB in the coordinates y, z). Moreover, the estimates in (2) hold

with $\lambda = \lambda_-$ and $\mu = \mu_-$.

We can generalize the above to the following setup: let $B = Z \times F$ be the product of two compact Riemannian manifolds; F and B may be manifolds with boundary. We suppose that $h : B \to B$ is a skew product itself, i.e. there exists a C^2-morphism $\zeta : Z \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{h} & B \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ Z & \xrightarrow{\zeta} & Z, \end{array} \tag{15}$$

where π is the standard projection. We assume that the map ζ downstairs is expanding, and that the fibers $\{z\} \times F$ are the stable manifolds of h:

$$\mu_- \leq |d\zeta^{-1}| \leq \mu, \tag{16}$$

$$\lambda_- \leq |dh| \leq \lambda \quad \text{on} \quad T(\{z\} \times F) \ , \forall z \in Z.$$

Again, for technical reasons we assume that $E^s = TF$ is trivialized as in (3). In this setup, Theorem 1 can be partially improved by the following result.

Theorem 2 *Consider a skew product $\mathcal{F}$ as above that also satisfies the modified dominated splitting condition. Then for small enough $\rho > 0$, any $\rho-$perturbation $\mathcal{G}$ of $\mathcal{F}$ has the following properties:*

a) There exists a continuous map $q : X \to Z$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathcal{G}} & X \\ {\scriptstyle q}\downarrow & & {\scriptstyle q}\downarrow \\ Z & \xrightarrow{\zeta} & Z \end{array} \tag{17}$$

commutes. Moreover, the commutative diagrams (7) and (17) must be compatible, in the sense that $q|_Y = \pi \circ p$.

b)The fibers $W^s_z = q^{-1}(z)$ are Lipschitz close to the vertical (constant) fibers, and Hölder continuous in z. This means that W^s_z is the graph of a Lipschitz map $\beta^s_z : F \times M \to Z$ such that

$$d(\beta^s_z, z)_{C^0} \leq O(\rho), \quad Lip\ \beta^s_z \leq O(\rho) \tag{18}$$

$$d(\beta^s_z, \beta^s_{z'})_{C^0} \leq \frac{d(z,z')^{\alpha - O(\rho)}}{O(\rho)^\alpha}, \tag{19}$$

where $\alpha = \dfrac{\ln \mu}{\ln \mu_-}$.

As was mentioned above, a particularly important case in which the theorem applies is the Smale-Williams solenoid with $Z = S^1$, $F = D$ and h given by (14).

1.4 Fubini revisited

Let Σ^2_+ be the set of all sequences $\omega^+ = \omega_0\omega_1\omega_2\dots$ of zeroes and ones, infinite to the right with the $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure. It is known that for almost all such sequences ω^+, any finite word w of any length n is encountered within ω^+ with frequency exactly equal to 2^{-n}. We are interested in sequences for which this property fails. More precisely, given $\kappa > 0$ and w a finite word of length n, we say that ω^+ is $\kappa, w-$*atypical* if the sequence

$$a_k(\omega^+, w) := \frac{\text{number of occurences of } w \text{ among first } k \text{ digits of } \omega^+}{k} \tag{20}$$

has a limit point *outside* $[2^{-n} - \kappa, 2^{-n} + \kappa]$. The sequence ω defines a point $y = \overline{0, \omega^+} \in S^1$ written in base 2. We say that y is $\kappa, w-$*atypical* if ω^+ is $\kappa, w-$atypical. Let $A_{\kappa,w} \subset S^1$ be the set of atypical points. By the ergodic theorem, it has Lebesgue measure 0 in S^1.

Theorem 3 *Consider a skew product $\mathcal{F}$ over the solenoid map as in Theorem 2. Let $\zeta : Z \to Z$ be the duplication of a circle. For any word w and positive κ there exists ρ such that the following holds. Let $A_{\kappa,w}$ be the same as above. Then for any ρ perturbation $\mathcal{G}$ of $\mathcal{F}$ and q as in* (17)*, we have:*

$$\text{mes } q^{-1}(A_{\kappa,w}) = 0. \tag{21}$$

In other words, the union of $\kappa, w-$atypical fibers has Lebesgue measure 0 in X.

An analog of this theorem for perturbations of skew products over the Anosov diffeomorphism of a two-torus may be proved basing on a recent result of P. Saltykov [21].

1.5 Attractors with intermingled basins

The tools developed in this paper allow us to get a new proof of the following phenomenon discovered by I. Kan:

Theorem 4 *[[11], [15]] The set of maps of an annulus* $S^1 \times [0,1]$ *that have intermingled attracting basins is open in the set of all maps of the annulus into itself that keep the boundary invariant.*

Intermingled attracting basins means the following thing: the Milnor attractor of the maps mentioned in Theorem 4 consists of the two boundary circles, each one having an attracting basin which is dense and of positive Lebesgue measure.

In [11], [15] the theorem above is improved by:

The complement to the union of the attracting basins in the perturbed Kan example has Hausdorff dimension smaller than 2.

Theorem 4, in a slightly different form, was claimed in [14]; as far as we know, the first proof was obtained in [1]. The same tools also allow us to construct diffeomorphisms with intermingled attracting basins [10]; the phase space in this case is the product of a solid torus and a circle.

2 Rate of contraction of the graph transform map

In this section we prove statement a) of Theorem 1, and establish the rate of contraction of the graph transform map, see Lemma 1 below. There are two ways to prove statement a). The first one is to establish partial hyperbolicity of the skew product $\mathcal{F}$, and refer to the Hirsch-Pugh-Shub theory. This theory implies the semiconjugacy statement a), but gives no estimate of the rate of contraction of the graph transform map. The second way is to revisit the graph transform map and to prove simultaneously the fixed point theorem and the rate of contraction estimate for this map. This is done in the present section.

2.1 Laminations

Let B, h, Λ be as in Subsection 1.2. In the fibers of the bundles E^s and E^u we have the abstract Riemannian metric, while in the fibers of the trivial bundles $B \times \mathbb{R}^k$ and $B \times \mathbb{R}^l$ we have the standard Euclidean metric. The isomorphism φ^s of (3) implies that there exist k linearly independent sections of E^s. By applying Gram-Schimdt orthonormalization to these sections, it follows that there exist k orthonormal sections of E^s. Sending a fixed orthonormal basis of $\mathbb{R}^k$ to these orthonormal sections will give us a *metric-preserving* isomorphism $B \times \mathbb{R}^k \to E^s$, and it is this isomorphism that we will henceforth denote by φ^s. The same discussion applies to φ^u.

For any $\delta > 0$, we define $Q^s(\delta)$ and $Q^u(\delta)$ to be the open balls of radius δ around the origin of $\mathbb{R}^k$ and $\mathbb{R}^l$, respectively. The metric-preserving isomorphisms φ^s and φ^u induce metric-preserving isomorphisms in each fiber:

$$\varphi_b^s(\delta) : Q^s(\delta) \to Q_b^s(\delta), \quad \varphi_b^u(\delta) : Q^u(\delta) \to Q_b^u(\delta), \tag{22}$$

where $Q_b^s(\delta) \subset E^s$ and $Q_b^u(\delta) \subset E^u$ are the open balls of radius δ around the origin in the respective fibers.

The number δ must be chosen small enough such that for any $b \in B$, the exponential map gives us an open embedding $Q_b^s(\delta) \times Q_b^u(\delta) \hookrightarrow B$. We write $B_b(\delta)$ for the image of this map. Composing this embedding with the isomorphism $\varphi_b^s(\delta) \times \varphi_b^u(\delta)$ gives us an open embedding (coordinate chart):

$$\varphi_b(\delta) : Q^s(\delta) \times Q^u(\delta) \hookrightarrow B. \tag{23}$$

Let us take $C > \max(\lambda_-^{-1}, \mu_-^{-1})$, and consider the above constructions for radius $C\delta$. Then we can express the map $h : B \to B$ locally around b in the domain and around $h(b)$ in the target. Therefore, in coordinates given by the chart (23), the map h has the form:

$$h_b(\delta) = (\varphi_{h(b)}(C\delta))^{-1} \circ h \circ \varphi_b(\delta), \quad (h^{-1})_b(\delta) = (\varphi_b(C\delta))^{-1} \circ h^{-1} \circ \varphi_{h(b)}(\delta). \tag{24}$$

For various values of δ, the maps $h_b(\delta)$ will represent the same germ at 0, but will have different domains. Similarly, the maps $(h_b)^{-1}(\delta)$ and $(h^{-1})_b(\delta)$ are representatives of the same germ at 0, but have different domains.

Until Section 3, we will work with a single, fixed δ. Therefore, we will often write simply $Q^s, Q^u, Q^s_b, Q^u_b, B_b, \varphi^s_b, \varphi^u_b, \varphi_b, h_b, (h^{-1})_b$ for the notions introduced in the previous paragraphs. By (2) and the fact that diffeomorphisms (22) are metric-preserving, dh_b has block-diagonal form at 0:

$$dh_b(0) = \mathrm{diag}(A^u, A^s),$$

where $\lambda_- \le |A_s| \le \lambda$ and $\mu_- \le |A_u^{-1}| \le \mu$. Because the coordinate charts φ_b are smooth functions, we have the following estimate throughout $Q^s \times Q^u$:

$$||dh_b - \mathrm{diag}(A^u, A^s)||_{C^0} \le O(\delta). \tag{25}$$

Now consider another compact manifold M, as in the statement of Theorem 1. For any domain A and any mapping $\beta : A \to B$, we will denote by $\gamma(\beta)$ the map from A onto the graph:

$$\gamma(\beta) : A \to A \times B, \quad \gamma(\beta) : a \mapsto (a, \beta(a)) \in A \times B. \tag{26}$$

Statement a) of Theorem 1 provides a correspondence between leaves and base points, so it's about time we defined these. The leaves of center-stable, center-unstable and center foliations corresponding to $b \in \Lambda$ are represented by Lipschitz maps:

$$\beta^s_b : Q^s \times M \to Q^u, \quad \beta^u_b : Q^u \times M \to Q^s, \quad \beta_b : M \to Q^u \times Q^s. \tag{27}$$

Then we define the leaves to be simply the graphs of the Lipschitz maps, embedded in $B \times M$ via (23):

$$W^*_b = \mathrm{Im}(\varphi_b \times \mathrm{Id}) \circ \gamma(\beta^*_b) \tag{28}$$

Here and below, $*$ stands for s, u or blank space.

Intuitively, W^s_b denotes a center-stable leaf, W^u_b denotes a central-unstable leaf, while W_b denotes a central leaf. We will never consider strongly stable or unstable leaves.

We now define certain functional spaces $\mathcal{B}^*$ of maps β^*. These are, by definition, the spaces of Lipschitz maps (27) that satisfy the condition:

$$\max\left\{\|\beta^*\|_{C^0}, \frac{\mathrm{Lip}\ \beta^*}{D}\right\} \le \frac{\delta}{2} \tag{29}$$

Here, D is a constant that will be picked in the proof of Lemma 1. The norm on the spaces $\mathcal{B}^*$ will always be the C^0 norm, and will be denoted by $||\cdot||$.

Intuitively speaking, a central-stable, central-unstable or central *lamination* is a continuous assignment of leaves as b runs over Λ. Rigorously speaking, a lamination is a continuous map:

$$S^* : \Lambda \to \mathcal{B}^*. \tag{30}$$

The map S^* is completely determined by the continuous collection of maps $\beta_b^* = S^*(b)$, as b ranges over Λ. Equivalently, S^* is completely determined by the leaves W_b^* of these maps.

The space of continuous sections S^* as above is denoted by Γ^*. The norm in this space is again the C^0 norm:

$$\|S^*\| = \max_{b\in\Lambda} \|S^*(b)\|.$$

For any $\delta > 0$ small enough, the metric space Γ^* with the distance $\rho(S_1^*, S_2^*) = ||S_1^* - S_2^*||$ is complete. Indeed, if $\beta_n^* \to \beta^*$ and $\mathrm{Lip}\ \beta_n^* \leq D\delta/2$, then $\mathrm{Lip}\ \beta^* \leq D\delta/2$.

Now consider a map $\mathcal{G} : B \times M \to B \times M$, like in the setup of Theorem 1. A central-stable, central-unstable or central lamination is called $\mathcal{G}-$invariant if its leaves W_b^* satisfy:

$$\mathcal{G}(W_b^s) \subset W_{h(b)}^s, \quad W_{h(b)}^u \subset \mathcal{G}(W_b^u) \tag{31}$$

$$\text{or} \quad W_{h(b)} = \mathcal{G}(W_b). \tag{32}$$

These conditions can all be written in terms of the maps β_b^* defining these leaves, and thus in terms of laminations S^* themselves. This will be done in the beginning of Subsection 2.2.

Our plan for the proof of Statement $a)$ of Theorem 1 will be the following: we will use the graph transform method described in the following subsection to find $\mathcal{G}-$invariant central-stable and central-unstable laminations. Then the central lamination will be given by

$$W_b = W_b^s \cap W_b^u.$$

Property (32) will follow from (31), so the central lamination will also be invariant under $\mathcal{G}$. Once we have the central lamination, we will define

$$Y = \bigsqcup_{b\in\Lambda} W_b.$$

Sending W_b to b defines the desired projection map $p : Y \to \Lambda$ of (7). Then the $\mathcal{G}-$invariance of the central lamination is precisely equivalent to the commutativity of diagram (7). We will follow this plan in the next subsections.

2.2 The graph transform map

Here we will deal with the $* = s$ case only, since the $* = u$ case is treated similarly. After that, the central case will be treated as described above. We will introduce first a "pointwise" graph transform map:

$$\mathfrak{g}_b : \mathcal{B}^s \longrightarrow \mathcal{B}^s$$

that acts on single leaves, and then a "global" graph transform map:

$$\mathfrak{g} : \Gamma^s \longrightarrow \Gamma^s$$

that acts on entire laminations. In both cases, the geometric idea is the same: start with a map $\beta^s : Q^s \times M \longrightarrow Q^u$ as in (27). Take the corresponding leaf $W^s_{h(b)} \subset B \times M$, and take its inverse image under $\mathcal{G}$. The claim is that we obtain a different leaf $\overline{W}^s_b \subset B \times M$, corresponding to a map $\overline{\beta}^s : Q^s \times M \longrightarrow Q^u$. Then we define the graph transform map as:

$$\mathfrak{g}_b(\beta^s) = \overline{\beta}^s.$$

In other words, the graph transform is implicitly defined by the following relation:

$$\{\mathcal{G}^{-1}(\varphi_{h(b)}(x_s, \beta^s(x_s, m)), m)\} = \{(\varphi_b(x_s, \overline{\beta}^s(x_s, m)), m)\}. \qquad (33)$$

We will prove in the appendix that the above correctly defines $\overline{\beta}^s$ (in other words, that the Implicit Function Theorem applies). The above definition also works in families. For a lamination $S^s \in \Gamma^s$ with leaves that are graphs of $\beta^s = S^s(h(b))$, define its graph transform as:

$$\mathfrak{g}(S^s) = (\overline{S}^s),$$

where $\overline{S}^s(b) = \overline{\beta}^s$ is defined by relation (33).

Comparing with (31), we see that a lamination S^s is $\mathcal{G}-$invariant if and only if it is a fixed point of the graph transform map $\mathfrak{g}$. Therefore, to show that there exists a unique $\mathcal{G}-$invariant central-stable lamination, we will use the fixed point principle: it is enough to show that $\mathfrak{g}$ is well defined and contracting.

Lemma 1 *For ρ small enough and any $\mathcal{F}, \mathcal{G}$ as in Theorem 1, there exists $\delta = O(\rho)$ so that the graph transform $\mathfrak{g}$ maps Γ^s into itself and is contracting with Lipschitz constant $\mu + O(\delta)$. In other words, for any $S_0^s, S_1^s \in \Gamma^s$ we have:*

$$\|\mathfrak{g}(S_0^s) - \mathfrak{g}(S_1^s)\| \leq (\mu + O(\delta))\|S_0^s - S_1^s\|. \tag{34}$$

In the pointwise situation, for any $b \in \Lambda$, we claim that $\mathfrak{g}_b$ maps $\mathcal{B}^s$ into itself. Furthermore, for any $\beta_0^s, \beta_1^s \in \mathcal{B}^s$, we have:

$$\|\mathfrak{g}_b(\beta_0^s) - \mathfrak{g}_b(\beta_1^s)\| \leq (\mu + O(\delta))\|\beta_0^s - \beta_1^s\|. \tag{35}$$

Corollary 1 *For $\delta = O(\rho)$ small enough, the graph transform map $\mathfrak{g}$ has a unique fixed point in Γ^s.*

Proof The statements about the global graph transform immediately follow from the corresponding statements in the pointwise case. So let us start by proving that $\mathfrak{g}_b$ maps $\mathcal{B}^s$ to itself. Take $b \in \Lambda$, $\beta^s \in \mathcal{B}^s$ and let $\overline{\beta}^s = \mathfrak{g}_b(\beta^s)$. We need to prove that:

$$\|\overline{\beta}^s\| \leq \frac{\delta}{2}, \tag{36}$$

$$\text{Lip } \overline{\beta}^s \leq \frac{D\delta}{2}. \tag{37}$$

Recall that γ_β is the map of $Q^s \times M$ onto the graph of β^s, see (26). In the Appendix we prove that for any $\beta = \beta^s$ that satisfies (29), there exists a Lipshitz homeomorphism $G_{\bar{\beta},b} : Q^s \times M \to Q^s \times M$, see (63), such that

$$\overline{\beta}^s = \pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta^s) \circ G_{\overline{\beta},b}.$$

Here $\mathcal{G}_b = (\varphi_{h(b)} \times \mathrm{Id})^{-1} \circ \mathcal{G} \circ (\varphi_b \times \mathrm{Id})$. Note that

$$||\overline{\beta}^s|| \leq ||\pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta^s)||,$$

because the shift in the argument of the right hand side does not change the C^0 norm. Therefore, by (6), we have:

$$||\overline{\beta}^s|| \leq ||\pi_u \circ \mathcal{F}_b^{-1} \circ \gamma(\beta^s)|| + O(\rho) = ||\pi_u \circ (h^{-1})_b \circ \gamma(\beta^s)|| + O(\rho).$$

By (25), we can further estimate the above:

$$||\overline{\beta}^s|| \leq (\mu + O(\delta))||\beta^s|| + O(\rho).$$

Since $\mu < 1$ and $||\beta^s|| \leq \delta/2$, for appropriately chosen $\rho = O(\delta)$ the above can be made $\leq \delta/2$. This proves (36). As for (37), note that

$$\text{Lip } \overline{\beta}^s \leq \text{Lip } (\pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta^s)) \cdot \text{Lip } G_{\overline{\beta},b}. \tag{38}$$

We need to show that the right hand side of the above is $\leq D\delta/2$. It is enough to do this for β^s and $\overline{\beta}^s$ of class C^1, since these maps are dense in $\mathcal{B}^s$. In this C^1 case, we have:

$$d(\pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta^s)) = d(\pi_u \circ \mathcal{G}_b^{-1}) \circ \gamma(\beta^s) \cdot d\gamma(\beta^s) \leq$$

$$\leq \left[d(\pi_u \circ \mathcal{F}_b^{-1}) \circ \gamma(\beta^s) + O(\rho)\right] \cdot d\gamma(\beta^s) \leq$$

$$\leq \left[\begin{pmatrix} \frac{\partial \pi_u \circ h_b^{-1}}{\partial x_s} & \frac{\pi_u \circ \partial h_b^{-1}}{\partial x_u} & 0 \end{pmatrix} \circ \gamma(\beta^s) + O(\rho)\right] \cdot \begin{pmatrix} 1 & 0 \\ \frac{\partial \beta^s}{\partial x_s} & \frac{\partial \beta^s}{\partial m} \\ 0 & 1 \end{pmatrix} \leq$$

$$\leq \left[\begin{pmatrix} 0 & \mu & 0 \end{pmatrix} + O(\delta) + O(\rho)\right] \cdot \begin{pmatrix} 1 & 0 \\ \frac{\partial \beta^s}{\partial x_s} & \frac{\partial \beta^s}{\partial m} \\ 0 & 1 \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} \mu \cdot \frac{\partial \beta^s}{\partial x_s} & \mu \cdot \frac{\partial \beta^s}{\partial m} \end{pmatrix} + O(\delta) + O(\rho) \leq \mu \cdot \text{Lip } \beta^s + O(\delta), \tag{39}$$

since $\rho = O(\delta)$. Combining this estimate with Proposition 6 of the Appendix, we see that:

$$\text{Lip } \overline{\beta}^s \leq (\mu \cdot \text{Lip } \beta^s + O(\delta)) \cdot (L + O(\delta)) \cdot (1 + \text{Lip } \overline{\beta}^s).$$

Since Lip $\beta^s \leq D\delta/2$, the above gives us:

$$\text{Lip } \overline{\beta}^s \leq \frac{\mu L \cdot D\delta/2 + L \cdot O(\delta) + O(\delta^2)}{1 - \mu L \cdot D\delta/2 - L \cdot O(\delta) - O(\delta^2)}.$$

By assumption (5), we have $\mu L < 1$. Therefore, if we pick the constant D large enough (but still requiring that $D\delta << 1$), the right hand side of the above will be $\leq D\delta/2$. This proves (37).

Now that we have proved $\mathfrak{g}$ and $\mathfrak{g}_b$ to be well-defined, let us pass to proving (34) and (35). As we said before, the second inequality implies the first, so we will only prove the second one. As above, write $\overline{\beta}_0^s = \mathfrak{g}_b(\beta_0^s)$ and $\overline{\beta}_1^s = \mathfrak{g}_b(\beta_1^s)$. From (64), we see that:

$$||\overline{\beta}_0^s - \overline{\beta}_1^s|| \leq T_1 + T_2, \tag{40}$$

where:

$$T_1 = ||\pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta_0^s) \circ G_{\overline{\beta}_0,b} - \pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta_1^s) \circ G_{\overline{\beta}_0,b}||,$$

$$T_2 = ||\pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta_1^s) \circ G_{\overline{\beta}_0,b} - \pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta_1^s) \circ G_{\overline{\beta}_1,b}||.$$

As it will soon be clear, T_1 is the dominant term:

$$T_1 \leq \text{Lip } (\pi_u \circ \mathcal{G}_b^{-1}) \cdot ||\gamma(\beta_0^s) - \gamma(\beta_1^s)||.$$

The second factor in the right hand side is $\leq ||\beta_0^s - \beta_1^s||$. As for the first factor, we see that:

$$\text{Lip } (\pi_u \circ \mathcal{G}_b^{-1}) \leq \text{Lip } (\pi_u \circ \mathcal{F}_b^{-1}) + O(\rho) = \text{Lip } (\pi_u \circ h_b^{-1}) + O(\rho) \leq \mu + O(\rho).$$

Since $\rho = O(\delta)$, we conclude that:

$$T_1 \leq (\mu + O(\delta)) \cdot ||\beta_0^s - \beta_1^s||. \tag{41}$$

As for T_2, we see that:

$$T_2 \leq \text{Lip } (\pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta_1^s)) \cdot ||G_{\overline{\beta}_0,b} - G_{\overline{\beta}_1,b}||.$$

In (39), we saw that:

$$\text{Lip } (\pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta_1^s)) \leq O(\delta).$$

In Proposition 7 of the Appendix, we will prove that

$$||G_{\overline{\beta}_0,b} - G_{\overline{\beta}_1,b}|| \le O(1) \cdot ||\overline{\beta}_0^s - \overline{\beta}_1^s||.$$

Therefore, we obtain:

$$T_2 \le O(\delta) \cdot ||\overline{\beta}_0^s - \overline{\beta}_1^s||.$$

Together with (41), this implies:

$$||\overline{\beta}_0^s - \overline{\beta}_1^s|| \le (\mu + O(\delta)) \cdot ||\beta_0^s - \beta_1^s|| + O(\delta) \cdot ||\overline{\beta}_0^s - \overline{\beta}_1^s|| \Rightarrow$$

$$||\overline{\beta}_0^s - \overline{\beta}_1^s|| \le (\mu + O(\delta)) \cdot ||\beta_0^s - \beta_1^s||.$$

This is precisely the desired inequality (35). □

2.3 The central lamination

Corollary 1 tells us that there exists a unique $\mathcal{G}-$invariant central stable lamination $S^s \in \Gamma^s$. This can be presented either via the maps β_b^s, or via the leaves W_b^s (as b ranges over Λ). Similarly, there exists a unique $\mathcal{G}-$invariant central unstable lamination $S^u \in \Gamma^u$. Let us define the central lamination S via its leaves W_b, which we define by:

$$W_b = W_b^s \cap W_b^u. \tag{42}$$

This lamination will be $\mathcal{G}-$invariant, in the sense of (32). Let us describe W_b more explicitly. By definition,

$$W_b^s = \text{Im } (\varphi_b \times \text{Id})\{(x_s, \beta_b^s(x_s, m), m)|\ x_s \in Q^s, m \in M\}$$

$$W_b^u = \text{Im } (\varphi_b \times \text{Id})\{(\beta_b^u(x_u, m), x_u, m)|\ x_u \in Q^u, m \in M\},$$

where β_b^s, β_b^u have Lipschitz norms at most $D\delta/2 << 1$. Then, for each $m \in M$, the system of equations

$$\begin{cases} x_s = \beta_b^u(x_u, m) \\ x_u = \beta_b^s(x_s, m) \end{cases} \tag{43}$$

has a unique solution $(x_s, x_u) =: \beta_b(m) \in Q^s \times Q^u$. Indeed, for any fixed m the maps $\beta_b^s \circ \beta_b^u : Q^u \to Q^u$ and $\beta_b^u \circ \beta_b^s : Q^s \to Q^s$ are Lipshitz with constant $\le (L\delta)^2 << 1$. Then each of the two maps is contracting and has a

unique fixed point: call these x_u and x_s, respectively. Then the pair (x_s, x_u) is the solution of (43), and the above map β_b is well-defined. If we define the map $\widetilde{\beta}_b = \varphi_b(\beta_b) : M \to B$, then its graph is precisely W_b:

$$W_b = \{(\widetilde{\beta}_b(m), m) |\ m \in M\}.$$

Because it is the intersection of an invariant central-stable lamination with an invariant central-unstable lamination, $S = (\beta_b) = (W_b)$ is an invariant central lamination.

It is not hard to see from (43) that

$$||\beta_b||_{C^0} \leq \frac{\delta}{2} \quad \text{and} \quad \frac{\text{Lip } \beta_b}{D} \leq \delta. \tag{44}$$

Because the chart φ_b is metric preserving at 0 and smooth in the domain $Q^s \times Q^u$ (which has diameter of order δ), we have:

$$d(\widetilde{\beta}_b, b)_{C^0} \leq O(\delta) = O(\rho), \qquad \text{Lip } \widetilde{\beta}_b \leq O(\delta) = O(\rho). \tag{45}$$

Proof of statement a) of Theorem 1: Start from the $\mathcal{G}-$invariant central lamination S constructed above, and define $Y = \bigcup_{b \in \Lambda} W_b$. This union is obviously an invariant set of $\mathcal{G}$, and moreover the following proposition implies that it is actually a disjoint union.

Proposition 1 *For all $b \neq b' \in \Lambda$, the corresponding central leaves are disjoint:*

$$W_b \cap W_{b'} = \emptyset.$$

Proof Let us assume by contraposition that $W_b \cap W_{b'} \neq \emptyset$. By the $\mathcal{G}-$invariance of the lamination, then

$$W_{h^k(b)} \cap W_{h^k(b')} \neq \emptyset,$$

for all $k \in \mathbb{Z}$. Pick a point $(\tilde{b}, m)$ in the above non-empty intersection. Then

$$\widetilde{b} = \widetilde{\beta}_{h^k(b)}(m) = \widetilde{\beta}_{h^k(b')}(m) \tag{46}$$

By (45), the point $\widetilde{\beta}_{h^k(b)}(m)$ is at distance at most $O(\rho)$ from $h^k(b)$. Similarly, $\widetilde{\beta}_{h^k(b')}(m)$ is at distance at most $O(\rho)$ from $h^k(b')$. This implies that $h^k(b)$ and

$h^k(b')$ are at most $2 \cdot O(\rho)$ apart, for all $k \in \mathbb{Z}$. This is obviously impossible for ρ small enough, because for such ρ, the quantity $O(\rho)$ is smaller than the expansivity constant of h.

□

Therefore, the map $p : Y \to \Lambda$ sending W_b to b is well-defined. Moreover, the $\mathcal{G}-$invariance of the lamination $S = (W_b)$ is precisely equivalent to the commutativity of the diagram (7). The continuity of p follows from the continuity of our laminations, and this also implies that the map H of (8) is continuous.

Note that the map H is bijective, with inverse given by $H^{-1}(b, m) = (\widetilde{\beta}_b(m), m)$. The map H^{-1} is clearly continuous in m, and continuity in b follows from the Hölder continuity statement (10), which will be proved in the next subsection. Therefore H is a homeomorphism, thus concluding the proof of statement a). □

3 Hölder continuity of the central lamination

This section will be concerned with the proof of statement b) of Theorem 1. By definition, we have $p^{-1}(b) = W_b = \text{Graph}(\widetilde{\beta}_b)$, where $\widetilde{\beta}_b$ satisfies relations (45). This is precisely the requirement (9). In this section we will prove the rest of statement b), which refers to Hölder continuity.

First, for any $b \in \Lambda$, we will define its *local central-stable* and *central-unstable manifolds* as

$$V_b^s = \{b' \in B | d(h^n(b'), h^n(b)) \leq \delta, \forall n \geq 0\},$$

$$V_b^u = \{b' \in B | d(h^{-n}(b'), h^{-n}(b)) \leq \delta, \forall n \geq 0\}.$$

Proposition 2 *Let h, Λ and d be the same as at the beginning of Subsection 1.2. Then the following statements hold for all $b, b' \in \Lambda$:*

1. if $b' \in V_b^s$ and $d(h^{-1}(b), h^{-1}(b')) \leq \delta \Rightarrow h^{-1}(b') \in V_{h^{-1}(b)}^s$

2. if $b' \in V_b^u$ and $d(h(b), h(b')) \leq \delta \Rightarrow h(b') \in V_{h(b)}^u$

3. if $b' \in V_b^s \Rightarrow \lambda_- - O(\delta) \leq \dfrac{d(h(b), h(b'))}{d(b, b')} \leq \lambda + O(\delta)$

4. if $b' \in V_b^u \Rightarrow \mu_- - O(\delta) \leq \dfrac{d(h^{-1}(b), h^{-1}(b'))}{d(b, b')} \leq \mu + O(\delta)$

Proof Statements 1 and 2 follow immediately from the definitions of V_b^s, V_b^u. We will now prove Statements 3 and 4. The map h has invariant stable and unstable laminations; V_b^s, V_b^u are the leaves of these laminations. They are smooth manifolds, and $V_b^s(V_b^u)$ is tangent at b to $E^s(E^u)$. Now Statements 3 and 4 follow from (2) and the C^2-smoothness of h. □

We further ask that h has the following *local product structure*: for all $b, b' \in \Lambda$ such that $d(b, b') \leq \delta$, there exists a unique $b^* \in B$ such that

$$V_b^u \cap V_{b'}^s = \{b^*\}, \tag{47}$$

and moreover:

$$d(b, b^*) + d(b', b^*) \leq O(d(b, b')). \tag{48}$$

This property is easily seen to hold for linear Anosov diffeomorphisms of the torus, because then V_b^s and $V_{b'}^u$ are just straight lines that meet transversely under a fixed angle independent of b, b'. It also holds for the Smale-Williams solenoid, because then $V_b^s = \{y(b)\} \times D$ and $V_{b'}^u$ is a curve that intersects V_b^s transversely (such that the angle between V_b^s and $V_{b'}^u$ is separated from zero).

Proof of statement b) of Theorem 1: We have already proved the closeness property (9) in relation (45) above. As for the Hölder property (10), it is enough to prove it for b, b' which are at most δ apart. Indeed, for any $\alpha > 0$ and any b, b' with $d(b, b') > \delta$, we have by default:

$$d(h(b), h(b')) \leq C d^\alpha(b, b'), \tag{49}$$

where $C = \frac{\text{diam}_B}{\delta^\alpha}$. Therefore, we can restrict attention to b, b' that are such that the unique point b^* of (47) satisfies:

$$d(b, b^*) \leq \delta, \ d(b', b^*) \leq \delta, \ d(b, b') \leq \delta, \tag{50}$$

For such nearby b, b', we essentially need to estimate the distance between the maps $\widetilde{\beta}_b, \widetilde{\beta}_{b'} : M \to B$. These maps were defined by the condition that

their graphs coincide with $W_b^s \cap W_b^u$ and $W_{b'}^s \cap W_{b'}^u$, respectively.

However, this is a bit of an issue: *different* leaves W_b^s and $W_{b^*}^s$ (and also their u counterparts) are defined using *different* coordinate charts φ_b and φ_{b^*}. To resolve this problem in the s-case (the u-case is treated in the same way), let us write W_b^s as the graph of a function $\overline{\beta}_b^s : Q^s \times M \to Q^u$ in the coordinate chart φ_{b^*}:

$$\{(\varphi_b \times \mathrm{Id}_M)(x_s, \beta_b^s(x_s, m), m)\} \overset{\mathrm{def}}{=} W_b^s = \{(\varphi_{b^*} \times \mathrm{Id}_M)(x_s, \overline{\beta}_b^s(x_s, m), m)\}$$

The function $\overline{\beta}_b^s$ is defined uniquely and implicitly by the above relation, but we must require the inclusion $\mathrm{Im}(\varphi_b) \subset \mathrm{Im}(\varphi_{b^*})$. We certainly cannot ensure this if we define the charts φ_b and φ_{b^*} with respect to the same δ in (23). But if we define φ_{b^*} with respect to 3δ instead of δ (i.e. define the chart on a neighborhood 3 times bigger), then the desired inclusion becomes a consequence of (50).

Definition 4 *With the assumption* (50)*, we define the distance between the leaves corresponding to* d, b^**, to be*

$$d(W_b^s, W_{b^*}^s) := ||\beta_{b^*}^s - \overline{\beta}_b^s||.$$

Implicit in the definition is the fact that the right hand side only makes sense on the domain of $\overline{\beta}_b^s$, which as was said before, is contained in the domain of $\beta_{b^*}^s$. Note that the above definition is *not* symmetric in b and b^*.

Now we must look at what happens with these leaves under the graph transform. Take two leaves $W_{h(b)}^s$ and $W_{h(b^*)}^s$, given in the coordinate chart $\varphi_{h(b^*)}$ by maps $\overline{\beta}_{h(b)}^s$ and $\beta_{h(b^*)}^s$, respectively. Then take their images under the graph transform W_b^s and $W_{b^*}^s$, given in the coordinate chart φ_{b^*} by maps $\overline{\beta}_b^s$ and $\beta_{b^*}^s$. The inequality (35) of Lemma 1 precisely says that:

$$d(W_b^s, W_{b^*}^s) \leq (\mu + O(\delta)) \cdot d(W_{h(b)}^s, W_{h(b^*)}^s). \tag{51}$$

Doing the analogous computations for central-unstable foliations, we see that:

$$d(W_{b'}^u, W_{b^*}^u) \leq (\lambda + O(\delta)) \cdot d(W_{h^{-1}(b')}^u, W_{h^{-1}(b^*)}^u). \tag{52}$$

Now recall that we fixed points b, b' satisfying relation (50). Let us consider the positive integers:

$$k = \left\lfloor \log_{\mu_- - O(\rho)} \frac{d(b, b^*)}{\delta} \right\rfloor, \quad l = \left\lfloor \log_{\lambda_- - O(\rho)} \frac{d(b', b^*)}{\delta} \right\rfloor.$$

Iterate relation (51) k times, and we obtain:

$$d(W^s_b, W^s_{b^*}) \leq (\mu + O(\delta))^k \cdot d(W^s_{h^k(b)}, W^s_{h^k(b^*)}).$$

By the definition of k (and property 4 of Proposition 2), k is the biggest positive integer which would ensure that the points $h^k(b)$ and $h^k(b^*)$ remain at most distance δ apart. Indeed, if they were at a bigger distance apart, the entire discussion above would break down. But since the distance between $h^k(b)$ and $h^k(b^*)$ is at most δ, we infer that the distance between the corresponding leaves is also at most $O(\delta)$. Therefore, the above inequality implies:

$$d(W^s_b, W^s_{b^*}) \leq (\mu + O(\delta))^k \cdot O(\delta) \leq d(b, b^*)^{\frac{\ln \mu}{\ln \mu_-} - O(\delta)} \cdot O(1).$$

The analogous discussion with l, λ, b', u instead of k, μ, b, s gives us:

$$d(W^u_{b'}, W^u_{b^*}) \leq (\lambda + O(\delta))^l \cdot O(\delta) \leq d(b', b^*)^{\frac{\ln \lambda}{\ln \lambda_-} - O(\delta)} \cdot O(1).$$

Letting α be defined as in (11), the above relations give us:

$$d(W^s_b, W^s_{b^*}) \leq d(b, b^*)^{\alpha - O(\delta)} \cdot O(1), \quad d(W^u_{b'}, W^u_{b^*}) \leq d(b', b^*)^{\alpha - O(\delta)} \cdot O(1). \tag{53}$$

Let's now prove that

$$W^s_{b'} \subset W^s_{b^*}, \quad \text{and analogously} \quad W^u_b \subset W^u_{b^*}. \tag{54}$$

Relation (51) for b replaced with b' becomes:

$$d(W^s_{b'}, W^s_{b^*}) \leq (\mu + O(\delta)) \cdot d(W^s_{h(b')}, W^s_{h(b^*)}).$$

However, since $b' \in V^s_{b^*}$, then the map h actually brings the points b' and b^* closer together (by property 3 of Proposition 2). So we can iterate the above inequalities as many times as we want. We see that:

$$d(W^s_{b'}, W^s_{b^*}) \leq (\mu + O(\delta))^i \cdot d(W^s_{h^i(b')}, W^s_{h^i(b^*)}),$$

for any $i > 0$. As $i \to \infty$, this implies $d(W^s_{b'}, W^s_{b^*}) = 0$. This proves (54) in the s-case. The proof in the u-case is similar.

We can now turn to the proof of (10), thus completing the proof of Theorem 1. Recall that for any $b \in B$, $W_b = W_b^s \cap W_b^u$. Let us first prove that

$$d(W_b, W_{b^*}) \le d(b, b_*)^{\alpha - O(\delta)} \cdot O(1),\ d(W_{b'}, W_{b^*}) \le d(b', b_*)^{\alpha - O(\delta)} \cdot O(1). \quad (55)$$

By (54) we have:

$$W_b = W_b^s \cap W_b^u,\ W_{b^*} = W_{b^*}^s \cap W_{b^*}^u. \quad (56)$$

In the chart $\varphi_{b^*} \times \mathrm{Id}$, the leaves $W_{b^*}^s$, $W_{b^*}^u$, W_b^s, W_b^u, $W_{b'}^s$, $W_{b'}^u$ are given by maps $\beta_{b^*}^s$, $\beta_{b^*}^u$, $\overline{\beta}_b^s$, $\overline{\beta}_b^u$, $\overline{\beta}_{b'}^s$, $\overline{\beta}_{b'}^u$. Then (53) gives us:

$$||\beta_{b^*}^s - \overline{\beta}_b^s||,\ ||\beta_{b^*}^u - \overline{\beta}_{b'}^u|| \le d(b, b')^{\alpha - O(\delta)} \cdot O(1),$$

while (54) gives us:

$$\beta_{b^*}^s = \overline{\beta}_{b'}^s,\ \beta_{b^*}^u = \overline{\beta}_b^u.$$

Of course, when one reads the above inequalities, one should keep in mind that the maps $\beta_{b^*}^{s,u}$ are defined on a neighborhood 3 times bigger than the maps $\overline{\beta}_{b,b'}^{s,u}$. Actually, the domain of the maps $\beta_{b^*}^{s,u}$ strictly contains the domain of the maps $\overline{\beta}_{b,b'}^{s,u}$. Therefore the above relations should be understood on the smaller domain, on which the maps $\overline{\beta}_{b,b'}^{s,u}$ are actually defined.

Now, relation (56) is equivalent to $\overline{\beta}_b(m) = (x_s, x_u)$ and $\beta_{b^*}(m) = (x_s^*, x_u^*)$, where:

$$\begin{cases} x_s = \overline{\beta}_b^u(x_u, m) \\ x_u = \overline{\beta}_b^s(x_s, m) \end{cases} \quad \begin{cases} x_s^* = \beta_{b^*}^u(x_u^*, m) \\ x_u^* = \beta_{b^*}^s(x_s^*, m) \end{cases} \quad (57)$$

For fixed m, the solutions (x_s, x_u) and (x_s^*, x_u^*) are fixed points of the contracting maps $\overline{\beta}_b^s \circ \overline{\beta}_b^u \times \overline{\beta}_b^u \circ \overline{\beta}_b^s : Q^s \times Q^u \to Q^s \times Q^u$ and $\beta_{b^*}^s \circ \beta_{b^*}^u \times \beta_{b^*}^u \circ \beta_{b^*}^s : Q^s \times Q^u \to Q^s \times Q^u$, respectively. The contraction coefficient is $<< 1$, uniformly in m and b. Therefore, the systems (57) have a unique solution for each m.

As was shown in (53) and (54), the maps $\beta_{b^*}^{s,u}$ and $\overline{\beta}_b^{s,u}$ of (57) are Hölder continuous in b. Therefore, the unique solutions of the systems (57) are also Hölder continuous in b, and thus so are the maps β_{b^*} and $\overline{\beta}_b$. Therefore, we have analogues of (53):

$$||\overline{\beta}_b - \beta_{b^*}|| \leq d(b,b^*)^{\alpha-O(\delta)} \cdot O(1), \quad ||\overline{\beta}_{b'} - \beta_{b^*}|| \leq d(b',b^*)^{\alpha-O(\delta)} \cdot O(1).$$

By the triangle inequality, this implies:

$$||\overline{\beta}_b - \overline{\beta}_{b'}|| \leq (d(b,b^*)^{\alpha-O(\delta)} + d(b',b^*)^{\alpha-O(\delta)}) \cdot O(1) \leq$$
$$\leq 2(d(b,b^*) + d(b',b^*))^{\alpha-O(\delta)} \cdot O(1) \leq d(b,b')^{\alpha-O(\delta)} \cdot O(1),$$

where the last inequality follows from (48). This proves the desired inequality in the chart $\varphi_{b^*} \times \mathrm{Id}$ (that is, for the maps $\overline{\beta}_b, \overline{\beta}_{b'} : M \longrightarrow Q^s \times Q^u$). On the manifold (that is, for the maps $\widetilde{\beta}_b, \widetilde{\beta}_{b'} : M \longrightarrow B$), the analogous relation follows from the fact that the derivative of φ_{b^*} at 0 is identity.

Therefore, relation (10) is proved. Note, that $O(1)$ above is a constant not depending on b, b', but depending on δ as in (49). We have put ρ in the denominator of (54) instead of δ, because $\rho = O(\delta)$. Finally, the inverse map H^{-1} of (8) is explicitly given as

$$H^{-1}(b,m) = (\widetilde{\beta}_b(m), m).$$

This map is Lipschitz in the variable m, and Hölder continuous in the variable b by (10). Therefore H^{-1} is Hölder continuous. This concludes the proof of Theorem 1. □

4 Hölder continuity of center-stable foliation

In this section we complete the proof of Theorem 2. Recall that in this theorem the map h is a skew product itself, see (15), whose fibers are globally defined stable manifolds for h. In this case, we will see that the central-stable leaves of $\mathcal{G}$ can also be globally defined.

By analogy with Subsection 2.1, for $z \in Z$ a *global central-stable leaf* is defined as a Lipschitz function

$$\beta^s_z : F \times M \to Z, \tag{58}$$

and its graph is defined as

$$W^s_z = \gamma(\beta^s_z) = \{(\beta^s_z(f,m), f, m) | (f,m) \in F \times M\}.$$

We ask that our leaves be Lipschitz close to the constant function z, in the sense that:

$$\max\left\{d(\beta_z^s, z)_{C^0}, \frac{\text{Lip } \beta_z^s}{D}\right\} \leq \frac{\delta}{2}. \tag{59}$$

Finally, a *global central-stable lamination* is defined as a continuous assignment $S^s = (\beta_z^s) = (W_z^s)$ of such leaves, as z ranges over Z. Such a lamination is called $\mathcal{G}-$invariant if

$$\mathcal{G}(W_z^s) = W_{\zeta(z)}^s, \ \forall z \in Z, \tag{60}$$

where D is so chosen that the estimates in the (sketch of the) proof below work out. All these constructions are analogous to the ones in Subsection 2.1. Moreover, the entire machinery of Lemma 1 applies to our situation and produces a unique $\mathcal{G}-$invariant lamination $S^s = (\beta_z^s)$ satisfying (59) for D properly chosen. We will henceforth focus solely on this lamination. In particular, since ζ is expanding we obtain:

$$d(\beta_z^s, \beta_{z'}^s)_{C^0} \leq \frac{d(z,z')^{\alpha - O(\rho)}}{O(\rho)^\alpha}, \quad \text{where } \alpha = \frac{\ln \mu}{\ln \mu_-}. \tag{61}$$

This is proven in analogous fashion to statement b) of Theorem 1, which was proved in the previous Section.

Proof of Theorem 2:

Proposition 3 *The leaves* $W_z^s = \text{Graph}(\beta_z^s)$ *are disjoint and they cover the whole of* X*:*

$$X = \bigsqcup_{z \in Z} W_z^s.$$

Proof The fact that the leaves are disjoint is proven analogously to Proposition 1. As for their union being the whole of X, this is equivalent to the following claim: for any $z \in Z$ and $y \in F \times M$, there exists $\widetilde{z} \in Z$ such that $\beta_{\widetilde{z}}^s(y) = z$. Let us fix y and z, and prove this claim.

Fix a coordinate neighborhood of radius 2δ of z inside Z. Let $D(z,\delta)$, $D(z,2\delta)$, $S(z,\delta)$, $S(z,2\delta)$ be the balls/spheres centered at z of radii δ and 2δ in Z, respectively. The map $f : D(z,\delta) \to D(z,2\delta)$ given by $f(\widetilde{z}) = \beta_{\widetilde{z}}(y)$ is well-defined, because (59) implies that $d(f(\widetilde{z}), \widetilde{z}) \leq \delta/2$. Moreover, (61)

implies that the map f is continuous. Therefore, sliding points along a straight line segment gives us an isotopy between the identity map of $D(z, \delta)$ and f:

$$h_t(\tilde{z}, y) = (\tilde{z}, y) + t((\beta_{\tilde{z}}(y) - \tilde{z}), 0)$$

Because of $d(f(\widetilde{z}), \widetilde{z}) \leq \delta/2$, the image of the boundary sphere $h_t(S(z, \delta))$ never touches the center z during this isotopy. Therefore, the index of z with respect to the sphere $h_t(S(z, \delta))$ does not change during the isotopy. Therefore

$$z \in \mathrm{Im}(f) \Rightarrow \exists \widetilde{z} \text{ such that } \beta^s_{\widetilde{z}}(y) = z.$$

$\square$

By Proposition 3, the map $q : X \to Z$ given by sending W^s_z to z is well-defined. Moreover, the $\mathcal{G}-$invariance condition (60) implies that q makes the diagram (17) commute. Part b) of Theorem 2 follows immediately from (59) and (61).

Finally, let us prove the relation $q|_Y = \pi \circ p$. Take any point $b = (f, z) \in F \times Z$, and recall that we denote $z = \pi(b)$. If we take the map $\beta^s_{\pi(b)}$ defining the global lamination (see (58)), and restrict it to the δ neighborhood of $f \in F$, we obtain a map $\overline{\beta}^s_b$ as in (27). In other words restricting the leaves of the global lamination $W^s_{\pi(b)}$ produces a valid local lamination $\overline{W}^s_b$. Since the global lamination $W^s_{\pi(b)}$ is $\mathcal{G}-$invariant, it is easily seen that the local lamination $\overline{W}^s_b$ will also be $\mathcal{G}-$invariant.

But local laminations are unique, as proved in Corollary 1! Therefore, the local leaves $\overline{W}^s_b$ coincide with the central-stable leaves W^s_b of Section 2. By the very definition of $\overline{W}^s_b$, this implies that $W^s_b \subset W^s_{\pi(b)}$, for all b. Since $W_b \subset W^s_b$ by construction, we conclude that:

$$W_b \subset W^s_{\pi(b)}, \ \forall \ b.$$

Now take any point $x \in Y = \bigsqcup_{b \in \Lambda} W_b$, and assume $x \in W_b$. By the definition of p, we have $p(x) = b$. But the above inclusion implies that $x \in W^s_{\pi(b)}$, and then by the definition of q, we have $q(x) = \pi(b)$. This precisely amounts to saying that $q|_Y = \pi \circ p$. $\square$

5 Fubini regained

In this section we prove Theorem 3. Moreover, at the end of this Section we discuss the "weak ergodic theorem" that appears in [11].

5.1 Measure zero and incomplete Hausdorff dimension

Let us begin by recalling the concept of Hausdorff dimension, denoted henceforth by $\dim_H$.

Definition 5 *Let A be a subset of a Euclidean space. A* cover *U of A is a finite or countable collection of balls Q_j of radii r_j whose union contains F. The d-dimensional volume of U, denoted by $V_d(U)$, is defined as*

$$V_d(U) = \sum_j r_j^{d_j}.$$

The Hausdorff dimension of A is defined as the infimum of those d for which there exists a cover of A with arbitrarily small $d-$dimensional volume:

$$\dim_H\, A = \inf\{d | \forall \varepsilon > 0 \ \exists \text{ a cover } U \text{ of } A \text{ such that } V_d(U) < \varepsilon\}.$$

Note that a compact manifold of dimension d also has Hausdorff dimension d. The same holds for a set of a positive Lebesgue measure on the Riemannian manifold of dimension d. Theorem 3 immediately follows from the following two propositions:

Proposition 4 *Recall the general setup of Theorem 2. If $A \subset Z$ satisfies*

$$\dim_H\, A < \frac{\ln \mu}{\ln \mu_-} \cdot \dim\, Z,$$

then for ρ small enough, the set $q^{-1}(A)$ has Lebesgue measure 0 in X.

Now recall the particular setup of Theorem 3, which takes place over the solenoid map. Note that in this case we have $Z = S^1$ and $\mu_- = \mu = \frac{1}{2}$.

Proposition 5 *For any $\kappa > 0$ and finite word w, there exists $\varepsilon = \varepsilon(\kappa, w)$ such that the set $A_{\kappa,w}$ of Subsection 1.4 has Hausdorff dimension at most $1 - \varepsilon$.*

5.2 Saving Fubini: the proof of Proposition 4

We are in the more general setup of Theorem 2. Since $X = Z \times F \times M$, the classical Fubini theorem states that

$$\mathrm{mes}(q^{-1}(A) \cap Z \times \{x\}) = 0, \forall x \in F \times M \Rightarrow \mathrm{mes}(q^{-1}(A)) = 0.$$

So all we need to do is to show that for any fixed $x \in F \times M$, the intersection $q^{-1}(A) \cap Z \times \{x\}$ has measure 0 in Z. By the very definition of the map q of (17), this intersection is nothing but the set $\{\beta_z^s(x) | z \in A\} \subset Z$. Moreover, by statement b) of Theorem 2 the map

$$\varphi : Z \to Z, \quad \varphi(z) = \beta_z^s(x)$$

is Hölder continuous with exponent $\alpha = \frac{\ln \mu}{\ln \mu_-} - O(\rho)$. All that we need to prove is that the set $\varphi(A)$ has measure 0 in Z. The following general lemma will do the trick:

Lemma 2 (Falconer) *Let Z be any Riemannian manifold, and $A \subset Z$ a subset. If $\varphi : Z \to Z$ is a Hölder map with exponent α, then*

$$\dim_H \varphi(A) \leq \frac{\dim_H A}{\alpha}$$

The proof of this Lemma can be found in [4]; the proof is straightforward. The above Lemma and the assumptions of Proposition 4 imply that for small enough ρ, we will have $\dim_H \varphi(A) < \dim Z$. Therefore, $\varphi(A)$ has Lebesgue measure 0 in Z, and as we have seen above this implies that $q^{-1}(A)$ has Lebesgue measure 0 in X. This concludes the proof of Proposition 4.

5.3 Large deviations: the proof of Proposition 5

In this section, we must prove that for any $\kappa > 0$ and finite word w, the set $A_{\kappa,w} \subset S^1$ of Subsection 1.4 has Hausdorff dimension at most $1 - \varepsilon$. Call a finite word of length N a $\kappa, w-$atypical word if the frequency of appearances of w in that word is outside the interval $[2^{-n} - \kappa, 2^{-n} + \kappa]$. Obviously, if a sequence is $\kappa, w-$atypical then infinitely many of its initial parts will be $\kappa, w-$atypical words. Thus for any N_0, we have the following inclusion:

$$\{\kappa, w\text{−atypical sequences}\} \subset \bigcup^{N \geq N_0} \bigcup_{v \text{ is } \kappa,w\text{−atypical}}^{\text{length } v = N} \{\text{sequences starting with } v\}.$$

Looking at the points of S^1 that correspond in binary notation to these sequences, we have:

$$A_{\kappa,w} \subset \bigcup^{N\geq N_0} \bigcup^{\text{length } v=N}_{v \text{ is } \kappa,w-\text{atypical}} \{\text{ball of radius } 2^{-N} \text{ around } \overline{0.v}\}.$$

This produces a covering U of the set $A_{\kappa,w}$, as in Definition 5. Let us compute the $1-\varepsilon$ dimensional volume of this covering:

$$V_{1-\varepsilon}(U) \leq \sum_{N\geq N_0} 2^{-N(1-\varepsilon)} \cdot \#\{\kappa, w - \text{atypical words of length } N\}.$$

By Theorem 5 below, we can estimate the number of $\kappa, w-$atypical words of length N, thus obtaining

$$V_{1-\varepsilon}(U) \leq \sum_{N\geq N_0} 2^{-N(\nu-\varepsilon)} = \frac{2^{-N_0(\nu-\varepsilon)}}{1-2^{\varepsilon-\nu}}.$$

If we choose $\varepsilon < \nu$ and let $N_0 \to \infty$, the above expression can be made arbitrarily small. Therefore, the Hausdorff dimension of the set $A_{\kappa,w}$ is at most $1-\varepsilon$. This concludes the proof of Proposition 5 and of Theorem 3, modulo the following estimate:

Theorem 5 (Large Deviation Theorem, [24]) *There exists $\nu = \nu(\kappa, w)$ such that for any N greater than some N_0, the number of $\kappa, w-$ atypical words of length N is at most $2^{N(1-\nu)}$.*

5.4 Weak ergodic theorem

For the sake of completeness, we formulate here the weak ergodic theorem of [11], whose proof is very closely related to the above material.

The classical ergodic theorem claims that for a given ergodic map and a continuous function φ, the set of points for which the time average of φ either does not exist or is not equal to the space average of φ, has measure zero. Here we claim that, for the duplication of a circle, for any fixed continuous function $\varphi \in C(S^1)$ and any $\delta > 0$, the set of points for which the sequence of partial time averages of φ has a limit point that differs from the space average of φ by more than δ, has Hausdorff dimension smaller than 1. We expect that this theorem may be generalized to any ergodic hyperbolic map of a compact Riemannian manifold M^n with a smooth invariant measure.

Theorem 6 *Let ζ be the duplication of the circle $S^1 = \mathbb{R}/\mathbb{Z}$, given by $\zeta(y) = 2y$. Let $\varphi \in C(S^1)$ and $\delta > 0$ be given. The partial time averages of φ and its space average are defined as:*

$$\varphi_n(y) = \frac{1}{n}\sum_{i=0}^{n-1} \varphi(\zeta^i(y)), \quad I = \int_{S^1} \varphi.$$

Then the set

$$K_{\varphi,\delta} = \{y | \text{ the sequence } \varphi_n(y) \text{ has a limit point outside } [I-\delta, I+\delta]\}$$

has Hausdorff dimension smaller than 1.

A similar theorem for Anosov diffeomorphisms of the two-torus was proved recently by Saltykov [21].

6 Appendix

6.1 The graph transform map made explicit

Recall that the graph transform map $\mathfrak{g}_b : \mathcal{B}^s \to \mathcal{B}^s$ was defined by $\beta^s \to \overline{\beta}^s$, where:

$$\{\mathcal{G}^{-1}(\varphi_{h(b)}(x_s, \beta^s(x_s, m)), m)\} \supset \{(\varphi_b(x_s, \overline{\beta}^s(x_s, m)), m)\}.$$

We now want to turn this implicit definition into an explicit formula. Recall our notation $\gamma(\beta)$, under which the above becomes:

$$\text{Im } \mathcal{G}^{-1} \circ (\varphi_{h(b)} \times \text{Id}) \circ \gamma(\beta^s) \supset \text{Im } (\varphi_b \times \text{Id}) \circ \gamma(\overline{\beta}^s).$$

If we write $\mathcal{G}_b = (\varphi_{h(b)}(C\delta) \times \text{Id})^{-1} \circ \mathcal{G} \circ (\varphi_b(\delta) \times \text{Id})$ as in (24), then our relation takes the form:

$$\text{Im } \mathcal{G}_b^{-1} \circ \gamma(\beta^s) \supset \text{Im } \gamma(\overline{\beta}^s). \tag{62}$$

Write $\pi_u : Q^s \times Q^u \times M \to Q^u$ and $\pi_{sc} : Q^s \times Q^u \times M \to Q^s \times M$ for the standard projections, and define:

$$G_{\overline{\beta},b} = \pi_{sc} \circ \mathcal{G}_b \circ \gamma(\overline{\beta}^s) : Q^s \times M \to Q^s \times M. \tag{63}$$

Then (62) is equivalent to:

$$\pi_u \circ \mathcal{G}_b^{-1} \circ \gamma(\beta^s) \circ G_{\overline{\beta},b} = \overline{\beta}^s. \tag{64}$$

Proposition 6 *The composition* (63) *is well-defined and*

$$\text{Lip } G_{\overline{\beta},b} \leq (L + O(\delta)) \cdot (1 + \text{Lip } \overline{\beta}^s),$$

where L is the constant from Definition 2. A similar estimate holds in the central-unstable case.

Proof Define the composition

$$F_{0,b} = \pi_{sc} \circ \mathcal{F}_b \circ \gamma(0),$$

in analogy with (63), with $\mathcal{G}$ replaced by $\mathcal{F}$ and $\overline{\beta}^s$ replaced by the zero map $0 : Q^s \times M \to Q^u$. Since $d(\mathcal{G}, \mathcal{F})_{C^1} \leq \rho$, we see that:

$$\text{Lip } G_{\beta,b} \leq (\text{Lip } F_{0,b} + O(\rho)) \cdot (1 + \text{Lip } \overline{\beta}^s) \tag{65}$$

But one can simply unravel the definition of $F_{0,b}$ when $\mathcal{F}$ is a skew product, and obtain

$$F_{0,b}(x_s, m) = (\pi_s \circ h_b(x_s, 0), f_{(x_s,0)}(m)).$$

From this it is clear that

$$\text{Lip } F_{0,b} \leq L + O(\delta).$$

Recalling that we always choose $\delta = O(\rho)$, (65) implies:

$$\text{Lip } G_{\overline{\beta},b} \leq (L + O(\delta)) \cdot (1 + \text{Lip } \overline{\beta}^s).$$

$\square$

Proposition 7 *For any two central-stable leaves $\overline{\beta}_0^s, \overline{\beta}_1^s \in \mathcal{B}^s$, we have:*

$$||G_{\overline{\beta}_0,b} - G_{\overline{\beta}_1,b}|| \leq O(1) \cdot ||\overline{\beta}_0^s - \overline{\beta}_1^s||.$$

A similar result holds in the central-unstable case.

Proof We have:

$$||G_{\overline{\beta}_0,b} - G_{\overline{\beta}_1,b}|| = ||\pi_{sc} \circ \mathcal{G}_b \circ \gamma(\overline{\beta}_0^s) - \pi_{sc} \circ \mathcal{G}_b \circ \gamma(\overline{\beta}_1^s)|| \leq$$

$$\leq \text{Lip } (\pi_{sc} \circ \mathcal{G}_b) \cdot ||\gamma(\overline{\beta}_0^s) - \gamma(\overline{\beta}_1^s)|| \leq O(1) \cdot ||\overline{\beta}_0^s - \overline{\beta}_1^s||.$$

$\square$

6.2 Persistence of Hölder skew products

The second, independent technical result that we will prove concerns the setup of Theorem 1: we have a small ρ−perturbation $\mathcal{G}$ of the skew product $\mathcal{F}$ from Theorem 1. This theorem tells us that $\mathcal{G}$ is conjugated to a skew product G:

$$G(b,m) = (h(b), g_b(m)).$$

In this Subsection, we will prove formulas (12) and (13). To this end, from the very definition of G we have the following explicit formula for the fiber maps g_b:

$$g_b(m) = \pi_m(\mathcal{G}(\widetilde{\beta}_b(m), m)), \quad g_b^{-1}(m) = \pi_m(\mathcal{G}^{-1}(\widetilde{\beta}_{h(b)}(m), m)) \tag{66}$$

where $\pi_m : X = B \times M \to M$ is the standard projection. Obviously, we have

$$f_b(m) = \pi_m(\mathcal{F}(b,m)), \quad f_b^{-1}(m) = \pi_m(\mathcal{F}^{-1}(h(b), m)).$$

Since $d(\mathcal{G}^{\pm 1}, \mathcal{F}^{\pm 1})_{C^1} < \rho$, it follows from the above formulas that

$$d(g_b, f_b)_{C^1} \leq d(\mathcal{G}(\widetilde{\beta}_b(m), m), \mathcal{F}(b,m))_{C^1} \leq$$

$$\leq d(\mathcal{G}(\widetilde{\beta}_b(m), m), \mathcal{G}(b,m))_{C^1} + \rho \leq ||\mathcal{G}||_{C^1} \cdot d(\widetilde{\beta}_b, b)_{C^1} + \rho = O(\rho),$$

and similarly for $d(g_b^{-1}, f_b^{-1})_{C^1}$. This proves (12). As for the Hölder property, we have that

$$d(g_b, g_{b'})_{C^0} \leq ||\mathcal{G}||_{C^1} \cdot d(\widetilde{\beta}_b, \widetilde{\beta}_{b'})_{C^0} \leq O(d(b,b')^\alpha),$$

by (10). The statement concerning $d(g_b^{-1}, g_{b'}^{-1})_{C^0}$ is proved analogously, thus concluding the proof of (13).

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